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Stability of Gorenstein Subcategories in Extriangulated Length Categories

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Abstract: Let θ be a length function and $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$ be an extriangulated length category with enough θ-projective objects. We introduce the concepts of the Gorenstein subcategories $\mathcal{GP}(\theta)$ and $\mathcal{GP}^n(\theta)$ in $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$, and we prove $\mathcal{GP}^n(\theta)=\mathcal{GP}(\theta)$ for any $n\geqslant 1$. This result provides a characterization of the Gorenstein homological property in $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$.

Key words: extriangulated length categories; θ-Gorenstein projective objects; proper $\mathcal{P}(\theta)$-resolution

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0 Introduction

The central idea of Gorenstein homological algebra is to generalize classical projective, injective, and flat modules, thereby yielding Gorenstein projective, injective, and flat modules. Based on these objects, the authors of Refs. [1-2] developed a new theoretical framework that parallels classical homological algebra. Auslander and Bridger[1] introduced fundamental concepts such as the stable category and Gorenstein dimensions. Later, Enochs and Jenda[2] defined Gorenstein projective modules and Gorenstein injective modules over arbitrary rings. Subsequently, Sather-Wagstaff *et al*[3] generalized the properties of such modules to general abelian categories and proposed the central open question regarding the stability of Gorenstein categories.

In abelian categories, Huang[4] provided a positive answer to the above problem by constructing a proper $\mathcal{W}$-resolution. Yang and Wang[5] proved the stability of Gorenstein categories in triangulated categories using a similar construction. As important algebraic structures, triangulated categories and exact categories share many analogous homological properties. To further unify triangulated and exact categories, Nakaoka and Palu[6] introduced extriangulated categories. Subsequently, Hu *et al*[7] defined the concept of a proper class in an extriangulated category and investigated the relevant properties of ξ-Gorenstein projective objects. Hu *et al*[8] extended the study of Gorensteinness to extriangulated categories with a proper class of $\mathbb{E}$-triangles, establishing the stability of Gorenstein subcategories $\mathcal{GW}(\xi)$, thus unifying Huang's results[4] in abelian categories and Yang and Wang's results[5] in triangu-

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lated categories. Furthermore, Hu *et al*[9] also established the theory of Gorenstein homological dimensions in extriangulated categories, proving that the global ξ-Gorenstein projective dimension is equal to the global ξ-Gorenstein injective dimension. Hu *et al*[10] established the triangulated equivalence of two types of quotient categories via ideal quotients in extriangulated categories.

In recent years, He and Zhou[11] introduced the concept of homological systems (Θ-systems) in extriangulated categories. They unified relevant categorical systems and proved the uniqueness of Θ-projective systems determined by a Θ-system. Later, Wang *et al*[12] defined extriangulated length categories and provided a more refined framework for studying relative homological algebra. Zheng and Zhang[13] introduced the concepts of θ-Gorenstein projective objects and θ-Gorenstein projective dimensions in extriangulated length categories. Inspired by their work, we consider the following question: In the framework of extriangulated length categories, does the stability of Gorenstein subcategories still hold?

The main result of this paper gives an affirmative answer to this question: Let θ be a length function and $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$ be an extriangulated length category. If $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$ has enough θ-projective objects, then $\mathcal{GP}^n(\theta)=\mathcal{GP}(\theta)$ for any $n\geqslant 1$.

1 Preliminaries

Let us recall some basic definitions of extriangulated categories from Ref. [6].

Let $\mathcal{C}$ be an additive category equipped with an additive bifunctor

$$\mathbb{E}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to \mathrm{Ab}$$

where Ab is the category of abelian groups, and $\mathcal{C}^{\mathrm{op}}$ denotes the opposite category of $\mathcal{C}$. For any objects $A,C\in\mathcal{C}$, an element $\delta\in\mathbb{E}(C,A)$ is called an $\mathbb{E}$-extension. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta)=[A\xrightarrow{x}B\xrightarrow{y}C]$$

to any $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$. This $\mathfrak{s}$ is called a realization of $\mathbb{E}$ if it makes the diagrams commutative in Definition 2.9 of Ref. [6]. A triplet $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is called an extriangulated category if it satisfies the following conditions:

1) $\mathbb{E}\colon\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathrm{Ab}$ is an additive bifunctor;

2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$;

3) $\mathbb{E}$ and $\mathfrak{s}$ satisfy the compatibility conditions in Definition 2.12 of Ref. [6].

Next, we mainly review some definitions and basic properties of the extriangulated length categories.

Definition 1 (Definition 3.1 in Ref. [12]) We say a map $\theta\colon \mathrm{Iso}(\mathcal{C})\to\mathbb{N}$ is a length function on $\mathcal{C}$ if it satisfies the following conditions:

1) For any $X\in\mathcal{C}$, $\theta(X)=0$ if and only if $X\cong 0$;

2) For any $\mathbb{E}$-triangle

$$X\longrightarrow L\longrightarrow M\overset{\delta}{\dashrightarrow}$$

in $\mathcal{C}$, we have $\theta(L)\leqslant\theta(X)+\theta(M)$. In addition, if $\delta=0$, then $\theta(L)=\theta(X)+\theta(M)$.

A length function θ is said to be stable if it satisfies the condition $\theta(L)=\theta(X)+\theta(M)$ for any $\mathbb{E}$-triangle $X\longrightarrow L\longrightarrow M\dashrightarrow$ in $\mathcal{C}$.

Definition 2 (Definition 3.4 in Ref. [12]) Let θ be a length function on $\mathcal{C}$. An $\mathbb{E}$-triangle

$$X\xrightarrow{x}L\xrightarrow{y}M\overset{\delta}{\dashrightarrow}$$

is called θ-stable (or simply stable when θ is clear from the context) if $\theta(L)=\theta(X)+\theta(M)$. In this case, x is called a θ-inflation, y is called a θ-deflation, and δ is called a θ-extension. Such an $\mathbb{E}$-triangle is also called an $\mathbb{E}_\theta$-triangle. If there is no confusion, we denote x by the symbol $X\rightarrowtail L$ and y by $L\twoheadrightarrow M$.

Following Section 3.4 of Ref. [12], for any objects $A,C\in\mathcal{C}$, define

$$\mathbb{E}_\theta(C,A)=\{\delta\in\mathbb{E}(C,A)\mid\delta \text{ is a } \theta\text{-extension}\}.$$

We also denote by $\mathfrak{s}_\theta=\mathfrak{s}\big|_{\mathbb{E}_\theta}$ the restriction of $\mathfrak{s}$ to $\mathbb{E}_\theta$.

Lemma 1 (Lemma 3.6 in Ref. [12]) Let θ be a length function on $\mathcal{C}$. Consider Fig. 1, which consists of $\mathbb{E}$-triangles.

1) If f and g are θ-inflations, then each $\mathbb{E}$-triangle in Fig. 1 is stable;

2) If i and j are θ-deflations, then each $\mathbb{E}$-triangle in Fig. 1 is stable.

Definition 3 (Definition 3.7 in Ref. [12]) Let θ be

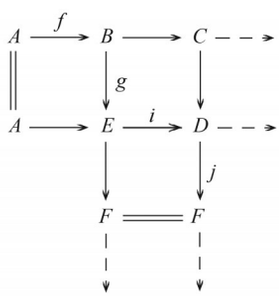


Fig. 1 A commutative diagram of $\mathbb{E}$-triangles

a length function on $\mathcal{C}$. A morphism $f: M \to N$ in $\mathcal{C}$ is θ-admissible if f admits a θ-decomposition (i_f, X_f, j_f), see Fig. 2, such that i_f is a θ-deflation and j_f is a θ-inflation.

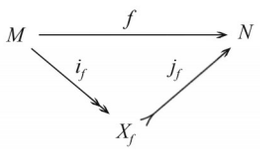


Fig. 2 θ-admissible

Definition 4 (Definition 3.8 in Ref. [12]) Let θ be a length function on $\mathcal{C}$. If all morphisms in $\mathcal{C}$ are θ-admissible, then $((\mathcal{C}, \mathbb{E}, \mathfrak{s}), \theta)$ is called an extriangulated length category or simply a length category. Sometimes, it is also abbreviated as $(\mathcal{C}, \theta)$ or simply $\mathcal{C}$.

Lemma 2 (Lemma 3.20 in Ref. [12]) Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be any composable pair of morphisms in $\mathcal{C}$. If the composite gf is a θ-inflation, then f is also a θ-inflation. Dually, if gf is a θ-deflation, then g is also a θ-deflation.

Lemma 3 (Lemma 3.21 in Ref. [12]) Let $\delta \in \mathbb{E}_\theta(A, B)$. For any $a: A' \longrightarrow A$ and $b: B \longrightarrow B'$, we have $a^*\delta \in \mathbb{E}_\theta(A', B)$ and $b_*\delta \in \mathbb{E}_\theta(A, B')$. In particular, $\mathbb{E}_\theta$ is an additive subfunctor of $\mathbb{E}$.

Proposition 1 (Proposition 3.22 in Ref. [12]) Let $((\mathcal{C}, \mathbb{E}, \mathfrak{s}), \theta)$ be an extriangulated length category. Then $((\mathcal{C}, \mathbb{E}_\theta, \mathfrak{s}_\theta), \theta)$ is a stable extriangulated length category. More precisely, $(\mathcal{C}, \mathbb{E}_\theta, \mathfrak{s}_\theta)$ is an extriangulated category, and θ is a stable length function with respect to this induced extriangulated structure.

Throughout the rest of this paper, unless otherwise stated, $((\mathcal{C}, \mathbb{E}, \mathfrak{s}), \theta)$ denotes an extriangulated length category. Every $\mathbb{E}_\theta$-triangle is regarded as an extriangle in the induced extriangulated category $(\mathcal{C}, \mathbb{E}_\theta, \mathfrak{s}_\theta)$. All constructions involving $\mathbb{E}_\theta$-triangles are in this induced extriangulated category.

2 Proper Resolutions and θ-Gorenstein Projective Objects

In this section, we recall the notions of θ-projective objects, proper $\mathcal{P}(\theta)$-resolutions, complete $\mathcal{P}(\theta)$-resolutions, and θ-Gorenstein projective objects. We also define the Gorenstein subcategory $\mathcal{GP}(\theta)$ and record several basic properties that will be used in Section 3. Throughout this section, all $\mathbb{E}_\theta$-triangles are regarded as extriangles in the induced extriangulated category $(\mathcal{C}, \mathbb{E}_\theta, \mathfrak{s}_\theta)$.

Definition 5 (Definition 3.5 in Ref. [13]) 1) An object $P \in \mathcal{C}$ is called θ-projective if for any $\mathbb{E}_\theta$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow,$$

the induced sequence

$$0 \longrightarrow \mathcal{C}(P, A) \longrightarrow \mathcal{C}(P, B) \longrightarrow \mathcal{C}(P, C) \longrightarrow 0$$

is exact.

We denote by $\mathcal{P}(\theta)$ the full subcategory of $\mathcal{C}$ consisting of all θ-projective objects. It is easy to verify that $\mathcal{P}(\theta)$ is an additive full subcategory, which is closed under isomorphisms and direct summands.

2) We say that the extriangulated length category $((\mathcal{C}, \mathbb{E}, \mathfrak{s}), \theta)$ has enough θ-projective objects if for any object $A \in \mathcal{C}$, there exists an $\mathbb{E}_\theta$-triangle $F \longrightarrow P \longrightarrow A \dashrightarrow$, with $P \in \mathcal{P}(\theta)$.

Throughout the rest of this paper, we assume that $((\mathcal{C}, \mathbb{E}, \mathfrak{s}), \theta)$ has enough θ-projective objects.

Definition 6 (Definition 3.7 in Ref. [13]) Let

$$\mathbf{X}: \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} \cdots$$

be an unbounded complex. The complex $\mathbf{X}$ is said to be θ-stable (or simply stable) if for every integer n, there exists an $\mathbb{E}_\theta$-triangle $G_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} G_n \dashrightarrow$ such that $d_n = g_{n-1}f_n$. The collection of $\mathbb{E}_\theta$-triangles is said to be induced by the complex $\mathbf{X}$.

Definition 7 (Definition 3.8 in Ref. [13]) Let $\mathcal{M}$ be a class of objects in $\mathcal{C}$. An $\mathbb{E}_\theta$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ is called $\mathcal{C}(-, \mathcal{M})$-exact (respectively $\mathcal{C}(\mathcal{M}, -)$-exact) if the induced sequence

$$0 \longrightarrow \mathcal{C}(C, M) \longrightarrow \mathcal{C}(B, M) \longrightarrow \mathcal{C}(A, M) \longrightarrow 0$$

(respectively

$$0 \longrightarrow \mathcal{C}(M, A) \longrightarrow \mathcal{C}(M, B) \longrightarrow \mathcal{C}(M, C) \longrightarrow 0)$$

is exact for any object $M \in \mathcal{M}$.

A θ-stable complex is called $\mathcal{C}(-, \mathcal{M})$-exact (respectively $\mathcal{C}(\mathcal{M}, -)$-exact), if every $\mathbb{E}_\theta$-triangle induced by the complex is $\mathcal{C}(-, \mathcal{M})$-exact (respectively $\mathcal{C}(\mathcal{M}, -)$-exact).

Definition 8 Let M be an object in $\mathcal{C}$. A $\mathcal{P}(\theta)$-resolution of M is an unbounded complex

$$\cdots \longrightarrow P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that $P_i \in \mathcal{P}(\theta)$ for every $i \geqslant 0$, and there exist objects $K_i \in \mathcal{C}$, with $K_0 = M$, together with $\mathbb{E}_\theta$-triangles

$$K_{i+1} \xrightarrow{g_i} P_i \xrightarrow{f_i} K_i \dashrightarrow, \ i \geqslant 0$$

satisfying $d_0 = f_0$ and $d_i = g_{i-1}f_i$ for every $i \geqslant 1$. The objects K_i are called the syzygies of the resolution.

A $\mathcal{P}(\theta)$-resolution of M is called a proper $\mathcal{P}(\theta)$-

resolution if every $\mathbb{E}_\theta$-triangle induced by the resolution is $\mathcal{C}(\mathcal{P}(\theta), -)$-exact. Dually, we can define a $\mathcal{P}(\theta)$-coresolution and coproper $\mathcal{P}(\theta)$-coresolution of M.

Remark 1 By Definition 5, every $\mathbb{E}_\theta$-triangle is automatically $\mathcal{C}(\mathcal{P}(\theta), -)$-exact. Therefore every $\mathcal{P}(\theta)$-resolution is proper.

Definition 9 (Definition 3.9 in Ref. [13]) A complex

$$\mathbf{P}: \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

is called a complete $\mathcal{P}(\theta)$-resolution if it satisfies the following conditions:

1) The complex $\mathbf{P}$ is stable;

2) All $\mathbb{E}_\theta$-triangles induced by the complex $\mathbf{P}$ are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact;

3) $P_n \in \mathcal{P}(\theta)$ for every $n \in \mathbb{Z}$.

Based on the preceding discussion, we now introduce θ-Gorenstein projective objects and the Gorenstein subcategory $\mathcal{GP}(\theta)$ of an extriangulated length category.

Definition 10 (Definition 3.10 in Ref. [13]) Let

$$\mathbf{P}: \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

be a complete $\mathcal{P}(\theta)$-resolution, and let

$$G_{i+1} \longrightarrow P_i \longrightarrow G_i \dashrightarrow, \ i \in \mathbb{Z}$$

be the $\mathbb{E}_\theta$-triangles induced by $\mathbf{P}$.

An object $G \in \mathcal{C}$ is called a θ-Gorenstein projective object if G is isomorphic to G_i for some complete $\mathcal{P}(\theta)$-resolution and some $i \in \mathbb{Z}$. We denote by $\mathcal{GP}(\theta)$ the full subcategory of $\mathcal{C}$ consisting of all θ-Gorenstein projective objects.

Definition 11 Let $\mathcal{W}$ be a full additive subcategory of $\mathcal{C}$ closed under isomorphisms. A complete $\mathcal{W}$-resolution of an object X is a complex

$$\cdots \longrightarrow W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} W_{-1} \longrightarrow \cdots$$

with $W_i \in \mathcal{W}$, together with $\mathbb{E}_\theta$-triangles

$$K_{i+1} \xrightarrow{u_i} W_i \xrightarrow{v_i} K_i \dashrightarrow, \ i \in \mathbb{Z}$$

such that $K_0 \cong X$, $d_i = u_{i-1} v_i$, and all these $\mathbb{E}_\theta$-triangles are both $\mathcal{C}(\mathcal{W}, -)$-exact and $\mathcal{C}(-, \mathcal{W})$-exact.

Denote by $\mathcal{G}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisting of all objects admitting a complete $\mathcal{W}$-resolution. By Definitions 9 and 10 and Remark 1,

$$\mathcal{G}\big(\mathcal{P}(\theta)\big) = \mathcal{GP}(\theta).$$

Set $\mathcal{GP}^1(\theta) = \mathcal{GP}(\theta)$, $\mathcal{GP}^{n+1}(\theta) = \mathcal{G}(\mathcal{GP}^n(\theta))$, $\forall n \geqslant 1$.

We now present some additional properties of the Gorenstein subcategory $\mathcal{GP}(\theta)$, which will be used in the proofs of subsequent results (see Theorem 1 and Theorem 2).

Theorem 1 (Theorem 3.13 in Ref. [13]) If $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ is an $\mathbb{E}_\theta$-triangle with $C \in \mathcal{GP}(\theta)$, then $A \in \mathcal{GP}(\theta)$ if and only if $B \in \mathcal{GP}(\theta)$.

Theorem 2 (Theorem 3.14 in Ref. [13]) The class $\mathcal{GP}(\theta)$ consisting of all θ-Gorenstein projective objects is closed under direct summands.

Remark 2 For any $\mathbb{E}_\theta$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ in $\mathcal{C}$, if $C \in \mathcal{P}(\theta)$, then this $\mathbb{E}_\theta$-triangle is split.

Lemma 4 (Lemma 3.12 in Ref. [13]) Let $\mathcal{M}$ be a subcategory of $\mathcal{C}$, and let

$$A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \dashrightarrow \text{ and } A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \dashrightarrow$$

be two $\mathbb{E}_\theta$-triangles. Consider Fig. 3.

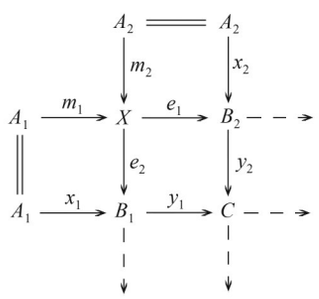


Fig. 3 The exactness of $\mathbb{E}_\theta$-triangles

If $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \dashrightarrow$ and $A_1 \xrightarrow{m_1} X \xrightarrow{e_1} B_2 \dashrightarrow$ are $\mathcal{C}(-, \mathcal{M})$-exact, then the sequence $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \dashrightarrow$ is $\mathcal{C}(-, \mathcal{M})$-exact.

Lemma 5 (Lemma 4.15 in Ref. [7]) Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \dashrightarrow$ be $\mathbb{E}_\theta$-triangles. Then the following conclusions hold:

1) If $(a, c): \delta \longrightarrow \delta'$ is a morphism of $\mathbb{E}_\theta$-triangles, where a and c are θ-inflations, then there is a θ-inflation $b: B \longrightarrow B'$, which yields Fig. 4;

2) Dually, if $(a, c): \delta \longrightarrow \delta'$ is a morphism of $\mathbb{E}_\theta$-triangles, where a and c are θ-deflations, then there is a θ-deflation $b: B \longrightarrow B'$, which yields the commutative diagram as in Fig. 4.

Proof By Proposition 1, $(\mathcal{C}, \mathbb{E}_\theta, \mathfrak{s}_\theta)$ is an extriangulated category. Moreover, it satisfies weakly idempotent completeness (WIC) (Condition 1 of Ref. [8]) by

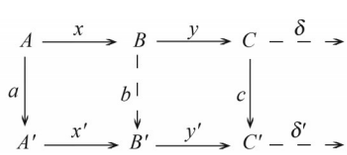


Fig. 4 A commutative diagram of θ-inflations

Lemma 2. Hence the result follows from Lemma 4.15 in Ref. [7] applied to this induced extriangulated category.

3 θ-Gorenstein Projective Objects in Extriangulated Length Categories

In this section, we provide a method to construct a proper $\mathcal{P}(\theta)$-resolution (and dually, coproper $\mathcal{P}(\theta)$-coresolution). Based on this construction, we further establish the stability of the Gorenstein subcategory $\mathcal{GP}(\theta)$ in extriangulated length categories.

The next lemma establishes the preservation of $\mathcal{C}(-, \mathcal{P}(\theta))$-exactness under the following commutative diagrams.

Lemma 6 Recall that $\mathcal{P}(\theta)$ denotes the full subcategory of θ-projective objects. Consider Fig. 5, which consists of $\mathbb{E}_\theta$-triangles. If the first row is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, then the second row is also $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

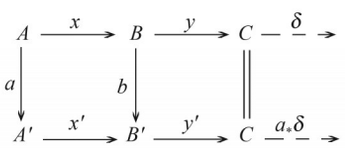


Fig. 5 The exactness of $\mathcal{C}(-, \mathcal{P}(\theta))$

Proof Let P be any object in $\mathcal{P}(\theta)$. We have the following exact sequence:

$$\mathcal{C}(C,P) \xrightarrow{\mathcal{C}(y',P)} \mathcal{C}(B',P) \xrightarrow{\mathcal{C}(x',P)} \mathcal{C}(A',P) \xrightarrow{(a_*\delta)^\#} \mathbb{E}(C,P),$$

where $(a_*\delta)^\#(f) = f_*a_*\delta$ for any $f \in \mathcal{C}(A',P)$.

Since $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, we have the following exact sequence:

$$0 \longrightarrow \mathcal{C}(C,P) \xrightarrow{\mathcal{C}(y,P)} \mathcal{C}(B,P) \xrightarrow{\mathcal{C}(x,P)} \mathcal{C}(A,P) \longrightarrow 0.$$

Since Fig. 5 is commutative, we have $y'b = y$. Hence, $\mathcal{C}(y,P) = \mathcal{C}(b,P)\mathcal{C}(y',P)$. Since $\mathcal{C}(y,P)$ is monic, it follows that $\mathcal{C}(y',P)$ is monic. For any morphism $f \in \mathcal{C}(A',P)$, we have $fa \in \mathcal{C}(A,P)$. Since $\mathcal{C}(x,P)$ is epic, there exists $g \in \mathcal{C}(B,P)$ such that $gx = fa$. Thus, $(a_*\delta)^\#(f) = f_*(a_*\delta) = (fa)_*\delta = (gx)_*\delta = g_*(x_*\delta) = 0$.

Therefore, $f \in \mathrm{Ker}(a_*\delta)^\#$. By $\mathrm{Ker}(a_*\delta)^\# = \mathrm{Im}\,\mathcal{C}(x',P)$, there exists a morphism $g' \in \mathcal{C}(B',P)$ such that $g'x' = f$, so $\mathcal{C}(x',P)$ is epic. Thus, $A' \xrightarrow{x'} B' \xrightarrow{y'} C \dashrightarrow$ is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Lemma 7 Consider morphisms of $\mathbb{E}_\theta$-triangles (Fig. 6).

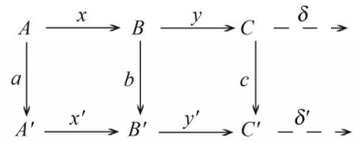


Fig. 6 A commutative diagram of morphisms between $\mathbb{E}_\theta$-triangles

Then the following conclusions hold:

1) If all morphisms of $\mathcal{C}(a,P)$, $\mathcal{C}(c,P)$, $\mathcal{C}(x',P)$ are epic for an object P in $\mathcal{P}(\theta)$, then so is $\mathcal{C}(b,P)$;

2) If all morphisms of $\mathcal{C}(a,P)$, $\mathcal{C}(c,P)$, $\mathcal{C}(y,P)$ are monic for an object P in $\mathcal{P}(\theta)$, then so is $\mathcal{C}(b,P)$;

3) If all morphisms of $\mathcal{C}(P,a)$, $\mathcal{C}(P,c)$, $\mathcal{C}(P,y)$ are epic for an object P in $\mathcal{P}(\theta)$, then so is $\mathcal{C}(P,b)$;

4) If all morphisms of $\mathcal{C}(P,a)$, $\mathcal{C}(P,c)$, $\mathcal{C}(P,x')$ are monic for an object P in $\mathcal{P}(\theta)$, then so is $\mathcal{C}(P,b)$.

Proof 1) Applying $\mathcal{C}(-,P)$ to Fig. 6, we obtain Fig. 7.

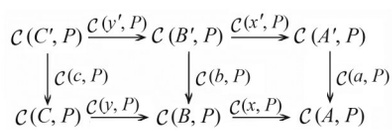


Fig. 7 The exactness of $\mathcal{C}(-,P)$

By the assumption, for any morphism $m \in \mathcal{C}(B,P)$, we have $mx \in \mathcal{C}(A,P)$. Since $\mathcal{C}(a,P)$ is epic, there exists $n \in \mathcal{C}(A',P)$ such that $na = mx \in \mathcal{C}(A,P)$. Similarly, since $\mathcal{C}(x',P)$ is epic, it can be concluded that there exists $e \in \mathcal{C}(B',P)$ such that $ex' \in \mathcal{C}(A',P)$, i.e., $ex' = n$. Thus, we obtain $mx = ex'a$. Furthermore, since $x'a = bx$, it follows that $mx = ebx$, i.e., $(m - eb)x = 0$. Then there exists a morphism $d \in \mathcal{C}(C,P)$ such that $dy = m - eb$. Thus, we have Fig. 8.

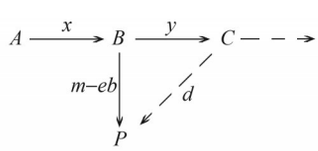


Fig. 8 The existence of a morphism d

It follows that $dy = m - eb$. Since $\mathcal{C}(c,P)$ is epic, it can be concluded that there exists $t \in \mathcal{C}(C',P)$ such that $tc = d \in \mathcal{C}(C,P)$. It follows that $tcy = m - eb$, $m = tcy + eb = ty'b + eb = (ty' + e)b$. Therefore, we conclude that $\mathcal{C}(b,P)$ is epic.

2) Similarly, consider Fig. 7. We prove that $\mathcal{C}(b,P)$ is monic, i. e., for any morphism $k \in \mathcal{C}(B',P)$, if $\mathcal{C}(b,P)(k) = 0$, then $k = 0$.

Suppose $k \in \mathcal{C}(B', P)$ with $\mathcal{C}(b, P)(k) = 0$, then $kb = 0 \in \mathcal{C}(B, P)$. We have $\mathcal{C}(x', P)(k) = kx' \in \mathcal{C}(A', P)$, $\mathcal{C}(a, P)(kx') = (kx')a \in \mathcal{C}(A, P)$.

Since $x'a = bx$, we have $(kx')a = k(x'a) = k(bx) = (kb)x = 0x = 0$. Because $\mathcal{C}(a, P)$ is monic, we obtain $kx' = 0$. Since $kx' = \mathcal{C}(x', P)(k) = 0$, we have $k \in \mathrm{Ker}\mathcal{C}(x', P) = \mathrm{Im}\mathcal{C}(y', P)$. Thus there exists $t \in \mathcal{C}(C', P)$ such that $k = \mathcal{C}(y', P)(t) = ty' \in \mathcal{C}(B', P)$. Substituting $k = ty'$ into $kb = 0$, we obtain $t(y'b) = 0$. Since $y'b = cy$, it follows that $t(y'b) = t(cy) = (tc)y = 0$, $\mathcal{C}(y, P)(tc) = 0$. Since $\mathcal{C}(y, P)$ is monic, $tc = 0$. Similarly, since $\mathcal{C}(c, P)(t) = 0$ and $\mathcal{C}(c, P)$ is monic, we have $t = 0$. Finally, $k = ty' = 0y' = 0$. Therefore, we conclude that $\mathcal{C}(b, P)$ is monic.

3) and 4) can be proved by analogy with 1) and 2).

The next proposition constructs commutative diagrams of $\mathbb{E}_\theta$-triangles in which all rows and columns are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Proposition 2 1) If $X \xrightarrow{f} Y \xrightarrow{f'} Z \dashrightarrow$ and $K \xrightarrow{g} L \xrightarrow{g'} Y \dashrightarrow$ are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles, then we have Fig. 9, where all rows and columns are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles.

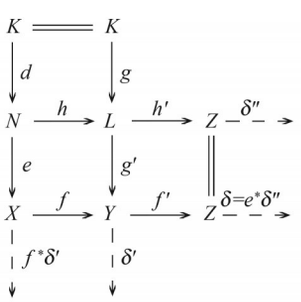


Fig. 9 An induced commutative diagram by f and g'

2) If $X \xrightarrow{f} Y \xrightarrow{f'} Z \dashrightarrow$ and $Y \xrightarrow{g} L \xrightarrow{g'} K \dashrightarrow$ are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles, then we have Fig. 10, where all rows and columns are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles.

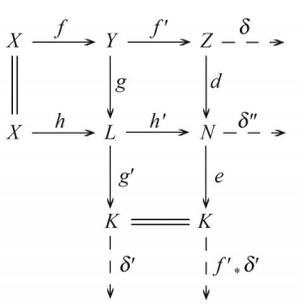


Fig. 10 An induced commutative diagram by f' and g

Proof 1) Applying the axiom (ET4)$^{\mathrm{op}}$ (Definition 2.12 in Ref. [6]) in the original extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to the two given $\mathbb{E}_\theta$-triangles, we obtain Fig. 9, in which all rows and columns are $\mathbb{E}$-triangles. We next show that all the $\mathbb{E}$-triangles in Fig. 9 are θ-stable.

Since $X \xrightarrow{f} Y \xrightarrow{f'} Z \dashrightarrow$ and $K \xrightarrow{g} L \xrightarrow{g'} Y \dashrightarrow$ are both $\mathbb{E}_\theta$-triangles, we have $\theta(Y) = \theta(X) + \theta(Z)$ and $\theta(L) = \theta(K) + \theta(Y)$.

To verify that $N \longrightarrow L \longrightarrow Z \dashrightarrow$ is an $\mathbb{E}_\theta$-triangle, we calculate: $\theta(L) = \theta(K) + \theta(Y) = \theta(K) + \theta(X) + \theta(Z) \geqslant \theta(N) + \theta(Z) \geqslant \theta(L)$. It follows that $\theta(L) = \theta(N) + \theta(Z)$. A similar calculation shows that $\theta(N) = \theta(K) + \theta(X)$. Thus, all the $\mathbb{E}$-triangles in Fig. 9 are stable. Consider Fig. 11.

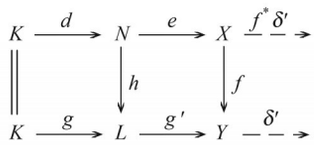


Fig. 11 The commutative diagram for Proposition 2

Since $X \longrightarrow Y \longrightarrow Z \dashrightarrow$ and $K \longrightarrow L \longrightarrow Y \dashrightarrow$ are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, it is easy to check that $\mathcal{C}(h, P)$ is epic by Lemma 7 1) for any object $P \in \mathcal{P}(\theta)$. $\mathcal{C}(h', P)$ is monic since $\mathcal{C}(h', P) = \mathcal{C}(g', P)\mathcal{C}(f', P)$. This implies that the second row is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact. It is easy to check that the first column is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact by 3×3-Lemma (Corollary 3.6 in Ref. [14]).

2) Dually, the proof follows as in 1).

Now, we start to construct new $\mathbb{E}_\theta$-triangles, which are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Lemma 8 Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be an $\mathbb{E}_\theta$-triangle. If both

$$K_1^A \xrightarrow{g_0^A} P_0^A \xrightarrow{f_0^A} A \dashrightarrow \text{ and } K_1^C \xrightarrow{g_0^C} P_0^C \xrightarrow{f_0^C} C \dashrightarrow$$

are $\mathbb{E}_\theta$-triangles with $P_0^A, P_0^C \in \mathcal{P}(\theta)$, then we have Fig. 12, where all rows and columns are $\mathbb{E}_\theta$-triangles. Moreover, if the third row and the first and third columns in Fig. 12 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, then all $\mathbb{E}_\theta$-triangles in Fig. 12 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Proof We first construct the underlying diagram of $\mathbb{E}$-triangles in extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Since $P_0^C \in \mathcal{P}(\theta)$, the homomorphism $\mathcal{C}(P_0^C, y): \mathcal{C}(P_0^C, B) \to \mathcal{C}(P_0^C, C)$ is epic. Hence there exists a morphism z: $P_0^C \to B$ such that $yz = f_0^C$. It follows that $(f_0^C)^*\delta = (yz)^*\delta = z^*(y^*\delta) = 0$.

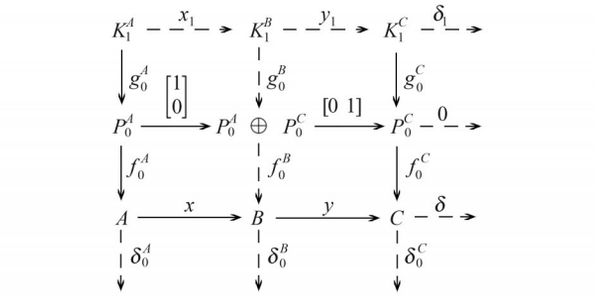


Fig. 12 An induced commutative diagram by f_0^A and f_0^C

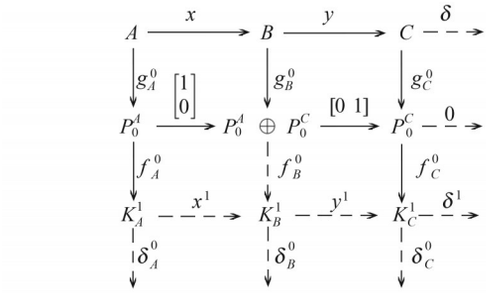


Fig. 13 An induced commutative diagram by f_A^0 and f_C^0

Consider the split $\mathbb{E}_\theta$-triangle

$$P_0^A \xrightarrow{\binom{1}{0}} P_0^A \oplus P_0^C \xrightarrow{(0\ 1)} P_0^C \dashrightarrow .$$

Since f_0^A and f_0^C are θ-deflations, Lemma 5 2) yields a θ-deflation $f_0^B: P_0^A \oplus P_0^C \longrightarrow B$, so there exists an $\mathbb{E}_\theta$-triangle $K_1^B \xrightarrow{g_0^B} P_0^A \oplus P_0^C \xrightarrow{f_0^B} B \dashrightarrow$. By Lemma 4.14 in Ref. [7], we have Fig. 12, in which all rows and columns are $\mathbb{E}$-triangles.

Note that

$$\begin{aligned}
&\theta(P_0^A)+\theta(P_0^C)\\
&=\theta(P_0^A\oplus P_0^C)\leqslant\theta(B)+\theta(K_1^B)\leqslant\theta(B)+\theta(K_1^A)+\theta(K_1^C)\\
&=\theta(B)+\theta(P_0^A)-\theta(A)+\theta(P_0^C)-\theta(C)\\
&=\theta(P_0^A)+\theta(P_0^C)+(\theta(B)-\theta(A)-\theta(C))\\
&=\theta(P_0^A)+\theta(P_0^C).
\end{aligned}$$

Hence we obtain $\theta(P_0^A\oplus P_0^C)=\theta(B)+\theta(K_1^B)$ and $\theta(K_1^B)=\theta(K_1^A)+\theta(K_1^C)$, it follows that the first row and the second column are $\mathbb{E}_\theta$-triangles. Combining with the assumptions, all rows and columns in Fig. 12 are $\mathbb{E}_\theta$-triangles.

Note that the $\mathbb{E}_\theta$-triangle $K_1^B \xrightarrow{g_0^B} P_0^A \oplus P_0^C \xrightarrow{f_0^B} B \dashrightarrow$ is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact by Lemma 7. Hence, the first row is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact by 3×3-Lemma (Corollary 3.6 in Ref. [14]). This completes the proof.

Similar to Lemma 8, we have the following:

Lemma 9 Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be a $\mathcal{C}(-,\mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangle. If both

$$A \xrightarrow{g_A^0} P_A^0 \xrightarrow{f_A^0} K_A^1 \dashrightarrow \text{ and } C \xrightarrow{g_C^0} P_C^0 \xrightarrow{f_C^0} K_C^1 \dashrightarrow$$

are $\mathbb{E}_\theta$-triangles with $P_A^0, P_C^0 \in \mathcal{P}(\theta)$, then we have Fig. 13, where all rows and columns are $\mathbb{E}_\theta$-triangles. If the first and third columns in Fig. 13 are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact, then all the $\mathbb{E}_\theta$-triangles in Fig. 13 are also $\mathcal{C}(-,\mathcal{P}(\theta))$-exact.

Proof The proof is dual to that of Lemma 8.

The next result provides a method for constructing a proper $\mathcal{P}(\theta)$-resolution of the first term in an $\mathbb{E}_\theta$-triangle from those of the last two terms.

Theorem 3 Let

$$X \xrightarrow{x} X^0 \xrightarrow{y} X^1 \dashrightarrow \tag{1}$$

be an $\mathbb{E}_\theta$-triangle. Assume that $\mathcal{P}(\theta)$ is closed under finite direct sums and cocones of θ-deflations, and let

$$\cdots \longrightarrow P_i^0 \xrightarrow{d_i^0} \cdots \longrightarrow P_1^0 \xrightarrow{d_1^0} P_0^0 \xrightarrow{d_0^0} X^0 \longrightarrow 0 \tag{2}$$

and

$$\cdots \longrightarrow P_i^1 \xrightarrow{d_i^1} \cdots \longrightarrow P_1^1 \xrightarrow{d_1^1} P_0^1 \xrightarrow{d_0^1} X^1 \longrightarrow 0 \tag{3}$$

be proper $\mathcal{P}(\theta)$-resolutions of X^0 and X^1, respectively.

1) We obtain the following proper $\mathcal{P}(\theta)$-resolution of X

$$\cdots \longrightarrow P_{i+1}^1 \oplus P_i^0 \longrightarrow \cdots \longrightarrow P_2^1 \oplus P_1^0 \longrightarrow P \longrightarrow X \longrightarrow 0 \tag{4}$$

and an $\mathbb{E}_\theta$-triangle

$$P \longrightarrow P_1^1 \oplus P_0^0 \longrightarrow P_0^1 \dashrightarrow . \tag{5}$$

2) If the θ-stable complexes (2), (3) and the $\mathbb{E}_\theta$-triangle (1) are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact, then (4) is also $\mathcal{C}(-,\mathcal{P}(\theta))$-exact.

Proof 1) Since (2) and (3) are proper $\mathcal{P}(\theta)$-resolutions of X^0 and X^1, respectively, there exist $\mathbb{E}_\theta$-triangles $K_{i+1}^0 \xrightarrow{g_i^0} P_i^0 \xrightarrow{f_i^0} K_i^0 \dashrightarrow$ and $K_{i+1}^1 \xrightarrow{g_i^1} P_i^1 \xrightarrow{f_i^1} K_i^1 \dashrightarrow$ with $P_i^0, P_i^1 \in \mathcal{P}(\theta)$ for each integer $i \geqslant 0$, where $K_0^0 = X^0$ and $K_0^1 = X^1$. Hence, we have Fig. 14, in which all rows and columns are $\mathbb{E}_\theta$-triangles. Thus, by Lemma 8, we have Fig. 15, where all rows and columns are $\mathbb{E}_\theta$-tri-

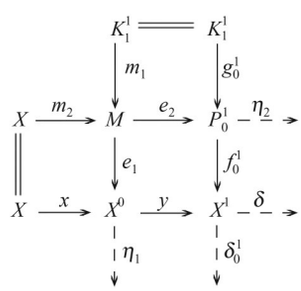


Fig. 14 A commutative diagram of M

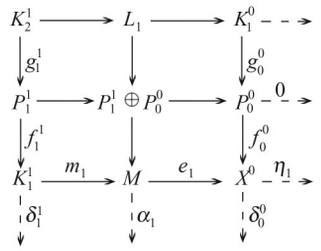


Fig. 15 A commutative diagram of L_1

angles and the second row is split. It is clear that $P_1^1 \oplus P_0^0 \in \mathcal{P}(\theta)$ by assumption. So we have Fig. 16, where all rows and columns are $\mathbb{E}_\theta$-triangles and the second row is the desired $\mathbb{E}_\theta$-triangle (5). Note that $P \in \mathcal{P}(\theta)$ because $\mathcal{P}(\theta)$ is closed under cocones of θ-deflations.

By Lemma 8 again, we have Fig. 17, where all rows and columns are $\mathbb{E}_\theta$-triangles. Iterating this process, we obtain the desired θ-stable complex (4), where $L_{i+1} \longrightarrow P_{i+1}^1 \oplus P_i^0 \longrightarrow L_i \dashrightarrow$, $\forall i \geqslant 1$, and $L_1 \longrightarrow P \longrightarrow X \dashrightarrow$ are $\mathbb{E}_\theta$-triangles.

2) The second row in Fig. 14 is split as $P_0^1 \in \mathcal{P}(\theta)$, so it is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact. Note that both the third row and the third column in Fig. 14 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, and hence $\mathcal{C}(e_1, P)$ is monic for any object $P \in \mathcal{P}(\theta)$ by Lemma 7 2). Since $\mathcal{C}(g_0^1, P) = \mathcal{C}(m_1, P)\mathcal{C}(e_2, P)$ is epic, it follows that $\mathcal{C}(m_1, P)$ is epic. Thus, it can be obtained that the second column in Fig. 14 is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact. Since both the first and third columns in Fig. 15 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, both the first row and the second column in this diagram are also $\mathcal{C}(-, \mathcal{P}(\theta))$-exact by Lemma 8. Since

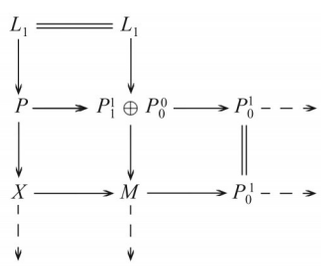


Fig. 16 A commutative diagram of P

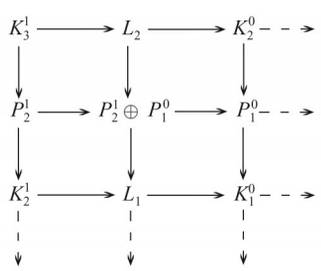


Fig. 17 A commutative diagram of L_2

both the third row and the second column in Fig. 16 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, it is easy to check that the first column in this diagram is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact by Lemma 8 and 3× 3-Lemma (Corollary 3.6 in Ref. [14]). Note that the third row, and the first and third columns in Fig. 17 are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, and hence the second column is also $\mathcal{C}(-, \mathcal{P}(\theta))$-exact by Lemma 8. Finally, we deduce that the θ-stable complex (4) is $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Dual to Theorem 3, we have the following result which provides a method for constructing a coproper $\mathcal{P}(\theta)$-coresolution of the last term in an $\mathbb{E}_\theta$-triangle from those of the first two terms.

Theorem 4 Let

$$Y_1 \longrightarrow Y_0 \longrightarrow Y \dashrightarrow \tag{6}$$

be an $\mathbb{E}_\theta$-triangle. Assume that $\mathcal{P}(\theta)$ is closed under finite direct sums and cones of θ-inflations, and let

$$0 \longrightarrow Y_0 \longrightarrow P_0^0 \longrightarrow P_0^1 \longrightarrow \cdots \longrightarrow P_0^i \longrightarrow \cdots \tag{7}$$

and

$$0 \longrightarrow Y_1 \longrightarrow P_1^0 \longrightarrow P_1^1 \longrightarrow \cdots \longrightarrow P_1^i \longrightarrow \cdots \tag{8}$$

be coproper $\mathcal{P}(\theta)$-coresolutions of Y_0 and Y_1, respectively. Then we obtain the coproper $\mathcal{P}(\theta)$-coresolution of Y

$$0 \longrightarrow Y \longrightarrow P \longrightarrow P_0^1 \oplus P_1^2 \longrightarrow \cdots \longrightarrow P_0^i \oplus P_1^{i+1} \longrightarrow \cdots \tag{9}$$

and an $\mathbb{E}_\theta$-triangle

$$P_1^0 \longrightarrow P_0^0 \oplus P_1^1 \longrightarrow P \dashrightarrow. \tag{10}$$

Proof The proof is dual to that of Theorem 3, so we omit it.

Theorem 5 Let

$$X_1 \longrightarrow X_0 \longrightarrow X \dashrightarrow \tag{11}$$

be an $\mathbb{E}_\theta$-triangle. Assume that $\mathcal{P}(\theta)$ is closed under finite direct sums, and let

$$P_0^n \longrightarrow \cdots \longrightarrow P_0^1 \longrightarrow P_0^0 \longrightarrow X_0 \longrightarrow 0 \tag{12}$$

and

$$P_1^{n-1} \longrightarrow \cdots \longrightarrow P_1^1 \longrightarrow P_1^0 \longrightarrow X_1 \longrightarrow 0 \tag{13}$$

be proper $\mathcal{P}(\theta)$-resolutions of X_0 and X_1, respectively.

1) Then we obtain the proper $\mathcal{P}(\theta)$-resolution of X:

$$P_0^n \oplus P_1^{n-1} \longrightarrow \cdots \longrightarrow P_0^2 \oplus P_1^1 \longrightarrow P_0^1 \oplus P_1^0 \longrightarrow P_0^0 \longrightarrow X \longrightarrow 0; \tag{14}$$

2) If the θ-stable complexes (12), (13) and the $\mathbb{E}_\theta$-triangle (11) are $\mathcal{C}(-, \mathcal{P}(\theta))$-exact, then the θ-stable complex (14) is also $\mathcal{C}(-, \mathcal{P}(\theta))$-exact.

Proof 1) Since (12) and (13) are proper $\mathcal{P}(\theta)$-resolutions of X_0 and X_1, respectively, there exist $\mathbb{E}_\theta$-triangles

$K_0^{i+1}\longrightarrow P_0^i\longrightarrow K_0^i\dashrightarrow$ and $K_1^{i+1}\longrightarrow P_1^i\longrightarrow K_1^i\dashrightarrow$ with $P_0^i, P_1^i\in\mathcal{P}(\theta)$ for each integer $i\geqslant 0$ with $K_0^0=X_0$ and $K_1^0=X_1$. Thus, we have Fig. 18, where all rows and columns are $\mathbb{E}_\theta$-triangles. Thus by Lemma 8, we have Fig. 19, where all rows and columns are $\mathbb{E}_\theta$-triangles. It is clear that $P_0^1\oplus P_1^0\in\mathcal{P}(\theta)$ by assumption. Then, by applying Lemma 8 again, we obtain the proper $\mathcal{P}(\theta)$-resolution (14) of X, which is spliced by $\mathbb{E}_\theta$-triangles:

$L_1\longrightarrow P_0^0\longrightarrow X\dashrightarrow$, $L_{i+1}\longrightarrow P_0^i\oplus P_1^{i-1}\longrightarrow L_i\dashrightarrow$, $\forall i\geqslant 1$.

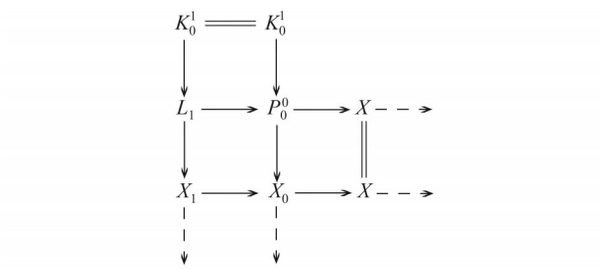


Fig. 18 An induced commutative diagram by L_1

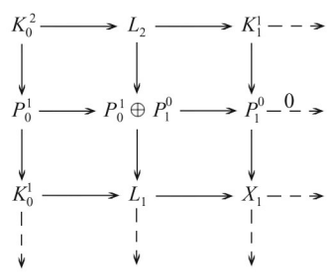


Fig. 19 An induced commutative diagram by L_2

2) It follows from Proposition 2 1) that all $\mathbb{E}_\theta$-triangles in Fig. 18 are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. Both the first and third column in Fig. 19 are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact by assumption, so the second column in this diagram is also $\mathcal{C}(-,\mathcal{P}(\theta))$-exact by Lemma 8. Finally, we deduce that the θ-stable complex (14) is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact.

The next result, which is dual to Theorem 5, provides a method for constructing a coproper $\mathcal{P}(\theta)$-coresolution of the first term in an $\mathbb{E}_\theta$-triangle from those of the last two terms.

Theorem 6 Let

$$Y\longrightarrow Y^0\longrightarrow Y^1\dashrightarrow \tag{15}$$

be a $\mathcal{C}(-,\mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangle. Assume that $\mathcal{P}(\theta)$ is closed under finite direct sums, and let

$$0\longrightarrow Y^0\longrightarrow P_0^0\longrightarrow P_1^0\longrightarrow\cdots\longrightarrow P_n^0 \tag{16}$$

and

$$0\longrightarrow Y^1\longrightarrow P_0^1\longrightarrow P_1^1\longrightarrow\cdots\longrightarrow P_{n-1}^1 \tag{17}$$

be coproper $\mathcal{P}(\theta)$-coresolutions of Y^0 and Y^1, respectively. Then we obtain the coproper $\mathcal{P}(\theta)$-coresolution of Y:

$$0\longrightarrow Y\longrightarrow P_0^0\longrightarrow P_0^1\oplus P_1^0\longrightarrow P_1^1\oplus P_2^0\longrightarrow\cdots$$
$$\longrightarrow P_{n-1}^1\oplus P_n^0. \tag{18}$$

Proof The proof is dual to that of Theorem 5, so we omit it.

As an important application of the above results, we obtain the main result of this paper.

Theorem 7 Let $((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$ be an extriangulated length category with enough θ-projective objects. Then we have $\mathcal{GP}^n(\theta)=\mathcal{GP}(\theta)$ for any $n\geqslant 1$.

Proof It suffices to show that $\mathcal{GP}^2(\theta)=\mathcal{GP}(\theta)$. Let $G\in\mathcal{GP}(\theta)$. Note that

$$G\xrightarrow{1}G\longrightarrow 0\dashrightarrow \text{ and } 0\longrightarrow G\xrightarrow{1}G\dashrightarrow$$

are $\mathbb{E}_\theta$-triangles. Note that

$$\cdots\longrightarrow 0\longrightarrow G\xrightarrow{1}G\longrightarrow 0\longrightarrow\cdots$$

is a complete $\mathcal{GP}(\theta)$-resolution of G, and then $G\in\mathcal{GP}^2(\theta)$. Indeed, all terms of this complex belong to $\mathcal{GP}(\theta)$, and all the induced $\mathbb{E}_\theta$-triangles are split. Hence they are both $\mathcal{C}(\mathcal{GP}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{GP}(\theta))$-exact. Therefore, $G\in\mathcal{GP}^2(\theta)$, which implies that $\mathcal{GP}(\theta)\subseteq\mathcal{GP}^2(\theta)$.

Let X be an object in $\mathcal{GP}^2(\theta)$ and

$$\cdots\longrightarrow G_1\longrightarrow G_0\longrightarrow G^0\longrightarrow G^1\longrightarrow\cdots$$

be a complete $\mathcal{GP}(\theta)$-resolution of X. There exist $\mathbb{E}_\theta$-triangles, which are both $\mathcal{C}(\mathcal{GP}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{GP}(\theta))$-exact:

$$K_{i+1}\longrightarrow G_i\longrightarrow K_i\dashrightarrow,\ K^i\longrightarrow G^i\longrightarrow K^{i+1}\dashrightarrow$$

with $G_i, G^i\in\mathcal{GP}(\theta)$ for any $i\geqslant 0$, where $K_0=K^0=X$.

First, we observe that $\mathcal{P}(\theta)\subseteq\mathcal{GP}(\theta)$. Indeed, let $P\in\mathcal{P}(\theta)$. Consider the split complex

$$\cdots\longrightarrow 0\longrightarrow P\xrightarrow{1}P\longrightarrow 0\longrightarrow\cdots.$$

All terms of this complex belong to $\mathcal{P}(\theta)$, and all the $\mathbb{E}_\theta$-triangles induced by the complex are split. Hence they are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. Moreover, by Remark 1, every $\mathbb{E}_\theta$-triangle is automatically $\mathcal{C}(\mathcal{P}(\theta),-)$-exact. Therefore, the above split complex is a complete $\mathcal{P}(\theta)$-resolution of P. Consequently, $P\in\mathcal{GP}(\theta)$. Since $P\in\mathcal{P}(\theta)$ was arbitrary, it follows that $\mathcal{P}(\theta)\subseteq\mathcal{GP}(\theta)$. Consequently, every $\mathbb{E}_\theta$-triangle which is both $\mathcal{C}(\mathcal{GP}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{GP}(\theta))$-exact is also both $\mathcal{C}(\mathcal{P}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{P}(\theta))$-exact.

Since $G_0\in\mathcal{GP}(\theta)$, there exists an $\mathbb{E}_\theta$-triangle,

which is both $\mathcal{C}(\mathcal{P}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{P}(\theta))$-exact: $L_1\longrightarrow P_0\longrightarrow G_0\dashrightarrow$ with $P_0\in\mathcal{P}(\theta)$ and $L_1\in\mathcal{GP}(\theta)$. Note that the $\mathbb{E}_\theta$-triangle $K_1\longrightarrow G_0\longrightarrow X\dashrightarrow$ is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact, and we have Fig. 20, where all rows and columns are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles by Proposition 2 1). Then, we have Fig. 21, where all rows and columns are $\mathbb{E}_\theta$-triangles.

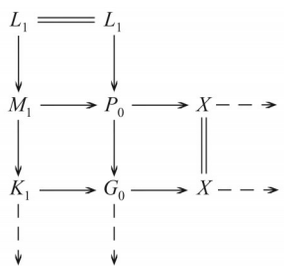


Fig. 20 A commutative diagram of M_1

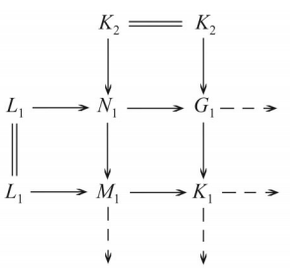


Fig. 21 A commutative diagram of N_1

Next we claim that the $\mathbb{E}_\theta$-triangle $L_1\longrightarrow N_1\longrightarrow G_1\dashrightarrow$ is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. There exists an $\mathbb{E}_\theta$-triangle, which is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact:

$$H\longrightarrow P\longrightarrow G_1\dashrightarrow$$

with $P\in\mathcal{P}(\theta)$ and $H\in\mathcal{GP}(\theta)$ since $G_1\in\mathcal{GP}(\theta)$. So, we have Fig. 22, where all rows and columns are $\mathbb{E}_\theta$-triangles in Fig. 22. Furthermore, because the second row in Fig. 22 is split as $P\in\mathcal{P}(\theta)$, it follows that it is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. Note that the third column in Fig. 22 is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. The third row $L_1\longrightarrow N_1\longrightarrow G_1\dashrightarrow$ is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact by Lemma 4. Since $G_1,L_1\in\mathcal{GP}(\theta)$, $N_1\in\mathcal{GP}(\theta)$ by Theorem 1. For every $P'\in\mathcal{P}(\theta)$, apply $\mathcal{C}(-,P')$ to Fig. 21. The third column is exact because it is induced by the complete $\mathcal{GP}(\theta)$-resolution and $\mathcal{P}(\theta)\subseteq\mathcal{GP}(\theta)$. The third row is exact by Fig. 20, and the second row has just been shown to be exact. Hence the second column is exact. Therefore, all rows and columns in Fig. 21 are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact.

Since $N_1\in\mathcal{GP}(\theta)$, there exists an $\mathbb{E}_\theta$-triangle which

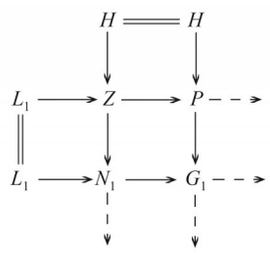


Fig. 22 A commutative diagram of Z

is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact: $L_2\longrightarrow P_1\longrightarrow N_1\dashrightarrow$ with $P_1\in\mathcal{P}(\theta)$ and $L_2\in\mathcal{GP}(\theta)$. So we have Fig. 23, where all rows and columns are $\mathbb{E}_\theta$-triangles that are $\mathcal{C}(-,\mathcal{P}(\theta))$-exact , by Proposition 2 1). So we can obtain $\mathcal{C}(-,\mathcal{P}(\theta))$-exact $\mathbb{E}_\theta$-triangles $M_{i+1}\longrightarrow P_i\longrightarrow M_i\dashrightarrow$ with $P_i\in\mathcal{P}(\theta)$. Splicing these $\mathbb{E}_\theta$-triangles together, we obtain the following θ-stable complex, which is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact

$$\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow X\longrightarrow 0$$

with $P_i\in\mathcal{P}(\theta)$, $i\geqslant 0$.

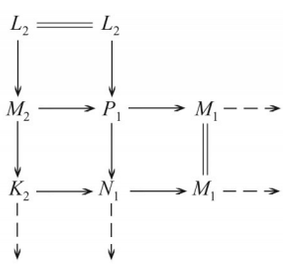


Fig. 23 A commutative diagram of M_2

Dually, we obtain a θ-stable complex that is $\mathcal{C}(-,\mathcal{P}(\theta))$-exact

$$0\longrightarrow X\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots$$

with $P^i\in\mathcal{P}(\theta)$, $i\geqslant 0$. We obtain a sequence

$$\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots,$$

where all its terms belong to $\mathcal{P}(\theta)$, and all its induced $\mathbb{E}_\theta$-triangles are both $\mathcal{C}(\mathcal{P}(\theta),-)$-exact and $\mathcal{C}(-,\mathcal{P}(\theta))$-exact. Hence, this is a complete $\mathcal{P}(\theta)$-resolution of X, that is, $X\in\mathcal{GP}(\theta)$. Thus, $\mathcal{GP}^2(\theta)\subseteq\mathcal{GP}(\theta)$. Therefore, we obtain $\mathcal{GP}^2(\theta)=\mathcal{GP}(\theta)$.

Inductively, $\mathcal{GP}^n(\theta)=\mathcal{GP}(\theta)$ for any $n\geqslant 2$. For n=1, it is obvious. Consequently, this proves the theorem.

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外三角长度范畴中Gorenstein子范畴的稳定性

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摘要：设θ是一个长度函数，$((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$为具有足够多$\theta$-投射对象的外三角长度范畴。我们引入了$((\mathcal{C},\mathbb{E},\mathfrak{s}),\theta)$中Gorenstein子范畴$\mathcal{GP}(\theta)$和$\mathcal{GP}^n(\theta)$的定义，并且证明了对于任意的$n\geqslant 1$，有$\mathcal{GP}^n(\theta)=\mathcal{GP}(\theta)$。该结论刻画了外三角长度范畴中的Gorenstein同调性质。

关键词：外三角长度范畴；θ-Gorenstein投射对象；真$\mathcal{P}(\theta)$-分解

□